4. Monotone Comparative Statics of Individual Choices

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MRes Microconomics

Overview

1. Monotone Comparative Statics
2. General Definitions
3. Monotone Comparative Statics of Individual Choices
4. More

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Some Motivation

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f: incentives, what motivates a particular kind of actions, environment

S: constraints, rules of the environment, legal system, etc.

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Comparative Statics: how changes in environment — in incentives f and/or constraints S — translate into changes in behaviour $X(S; f)$

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- Fine idea, but what is a higher maximiser when the alternatives are not real numbers?

  And what does a higher *set* of maximisers (or feasible set) mean? And what is an increase in the *function* f?

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   - Ordering Elements
   - Ordering Sets
   - Ordering Functions
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Throughout: $(X, \geq)$ **partially ordered set**

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- **Join** of $x, x' \in X$ taken wrt X: $x \vee_X x' := \inf\{y \in X : y \geq x \text{ and } y \geq x'\}$.
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  NB: inf and sup taken wrt $\geq$.

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Definition

(i) $(X, \geq)$ is a **lattice** iff it is a partially ordered set s.t. $\forall x, x' \in X$, $x \vee_X x' \in X$ and $x \wedge_X x' \in X$.
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Ordering Sets

Definition (Strong Set Order)

S' **strong set dominates** S (writing $S' \geq_{SS} S$) if $\forall x' \in S'$, $x \in S$, $x \vee x' \in S'$ and $x \wedge x' \in S$.

S' strong set dominates S if, taking any one element from each set, the join belongs to the dominating set and the meet to the dominated set.

In some cases, the strong set order can be too demanding.

Ordering Functions

Definition

Let $f : X \times T \to \mathbb{R}$, where X, T are partially ordered sets, and joins and meets of elements in $X \times T$ are wrt product order.

(i) f satisfies the **single-crossing property** (SCP) in $(x; t)$ if $\forall x, x' \in X$, $t, t' \in T$, s.t. $x' > x$ and $t' > t$, $f(x'; t) - f(x; t) \geq (>) \mathbf{0} \implies f(x'; t') - f(x; t') \geq (>) \mathbf{0}$.
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f is **submodular** if $-f$ is supermodular.

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NB: SM $\implies$ {QSM, ID} $\implies$ SCP.

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Behavioural Implications: With finite data, preference relations on a lattice have (a) a weakly monotone and quasisupermodular utility representation if and only if they have (b) a weakly monotone and supermodular utility representation (Chambers & Echenique 2009 JET)

Ordering Functions

Proposition

(i) If f and g are supermodular real-valued functions on X, then $\alpha f + \beta g$ are supermodular $\forall \alpha, \beta \geq 0$.

(ii) If $\exists$ strictly increasing $g : \mathbb{R} \to \mathbb{R}$ such that $g \circ f$ is supermodular, then f is quasisupermodular.

(iii) If $f \in \mathcal{C}^2$ in $y \in Y \equiv X \times T$, then f is supermodular in y if and only if $\frac{\partial^2}{\partial y_i \partial y_j} f \geq 0$, $\forall i \neq j$.

(iv) If X and Y are partially ordered sets, $X \times Y$ is a lattice with respect to the product order, and $f : X \times Y \to \mathbb{R}$ is supermodular, then $g(x) := \sup_{y \in Y} f(x, y)$ is supermodular.

(Proof left as an exercise.)

Ordering Functions

SM, QSM, ID, SCP are properties of *a* function.

But, we can think of $f(x, t)$ as defining a family of functions on X parametrised by t, $f_t(x) := f(x, t)$.

We can adjust definitions to order functions!

Definition

Let v, u be two real-valued functions on X; v **single-crossing** dominates u ($v \geq_{SC} u$) if $\forall x, x' \in X$ such that $x' \geq x$, $u(x') - u(x) \geq (>) \mathbf{0} \implies v(x') - v(x) \geq (>) \mathbf{0}$.

Overview

Strong Monotone Comparative Statics

Theorem (Milgrom & Shannon 1994 Theorem 4)

Let X be a lattice and v, u be two real-valued functions on X. v and u are quasisupermodular and v single-crossing dominates u if and only if, for $S' \geq_{SS} S$, $X(S'; v) \geq_{SS} X(S; u)$.

A very powerful result!

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Then $x \in X(S; u) \implies x \vee x' \in X(S'; v)$

and

$$\begin{aligned} & x' \in X(S'; v) \\ \implies & v(x \vee x') - v(x') \leq 0 \qquad \text{optimality of } x' \end{aligned}$$

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As $S' \geq_{ss} S$, we have $x \wedge x' \in S$ and $x \vee x' \in S'$.

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and

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\begin{aligned}
& x' \in X(S'; v) \\
\implies & v(x \vee x') - v(x') \leq 0 && \text{optimality of } x' \\
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Strong Monotone Comparative Statics

Theorem (Milgrom & Shannon 1994 Theorem 4)

Let X be a lattice and v, u be two real-valued functions on X. v and u are quasisupermodular and v single-crossing dominates u if and only if, for $S' \geq_{SS} S$, $X(S'; v) \geq_{SS} X(S; u)$.

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Hence $X(S'; v) \geq_{SS} X(S; u)$

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  Let $S = \{x, x'\}$ with $x' > x$.

  As $X(S; v) \geq_{ss} X(S; u)$, $x' \in (=) X(S; u) \implies x' \in (=) X(S; v)$.

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  As $X(S; v) \geq_{ss} X(S; u)$, $x' \in (=) X(S; u) \implies x' \in (=) X(S; v)$.

  And then, $u(x') - u(x) \geq (>) \mathbf{0} \implies v(x') - v(x) \geq (>) \mathbf{0}$. $\square$

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Two simple corollaries:

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Corollary 1 (Milgrom & Shannon 1994)

Let X be a lattice and $f : X \to \mathbb{R}$. f is quasisupermodular if and only if, for $S' \geq_{ss} S$, $X(S'; f) \geq_{ss} X(S; f)$.

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Proof

Take $v = u = f$. $\square$

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Corollary 2 (Milgrom & Shannon 1994)

Let X be a lattice, S a sublattice, and $f : X \to \mathbb{R}$. If f is quasisupermodular, then $X(S; f)$ is a sublattice of S.

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Proof

Take $v = u = f$ and $S = S'$. Hence $X(S; f) \geq_{ss} X(S; f)$. $\square$

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Theorem (Milgrom & Shannon 1994 Theorem 4')

Let X be a lattice and $v, u : X \to \mathbb{R}$. v and u are quasisupermodular and v strictly single-crossing dominates u then $\forall x' \in X(S'; v), x \in X(S; u), x' \geq x$.

any maximiser in $X(S'; v)$ is greater than *any* maximiser in $X(S; u)$!

Overview

1. Monotone Comparative Statics

2. General Definitions

3. Monotone Comparative Statics of Individual Choices

4. More

More

- Comparative statics in choice and risk and uncertainty: Athey (2002 QJE).
- Comparative statics of equilibrium outcomes: more later.
- Applications: *Macro:* Effects of changes in consumers' preferences on prices and output (Acemoglu & Jensen 2015 JPE). *Econometrics:* Nonparametric identification (Lazzati 2015 QE). *Health:* Junior doctors' residency programme matching (Agarwal 2015 AER)